11 Cauchy's integral formula and first applications

Holomorphic functions are remarkable creatures. For instance, the value of a holomorphic function inside a closed path is totally determined by the values of it along this path. To be more precise,

Proposition 11.1. *Let f be holomorphic in simply connected domain E. Then*

$$f(z)=\frac{1}{2\pi \mathrm{i}}\oint_{\gamma}\frac{f(w)}{w-z}\mathrm{d}w$$

for any point z inside positively oriented closed path $\gamma\subseteq E$.

Proof. First note that now we need to be careful about the orientation. Second, by Cauchy's theorem and the assumption that E is simply connected I can reduce my integral from $F(w)=f(w)/(w-z)$ to the integral along $\gamma_\epsilon=\partial B(z,\epsilon)$ along a circle with center at z of radius ϵ:

$$\int_{\gamma}F=\int_{\gamma_\epsilon}F.$$

The last integral can be written as

$$\int_{\gamma_\epsilon}\frac{f(w)}{w-z}\mathrm{d}z=\int_{\gamma_\epsilon}\frac{f(w)-f(z)}{w-z}\mathrm{d}w+\int_{\gamma_\epsilon}\frac{f(z)}{w-z}\mathrm{d}w.$$

The second integral in the last line is our *fundamental* example, and evaluates to $2\pi\mathrm{i}f(z)$. All I need to do is to show that the first term in the sum approaches zero as $\epsilon\to 0$. To this end I note that since f is differentiable at z, the ratio

$$\frac{f(w)-f(z)}{w-z}$$

is bounded for all w in $B(z,\epsilon)$ and therefore bounded on $\partial B(z,\epsilon)$. It implies, by ML-inequality, that

$$\left|\int_{\gamma_\epsilon}\frac{f(w)-f(z)}{w-z}\mathrm{d}w\right|\leq M\cdot 2\pi\epsilon\to 0,\quad \epsilon\to 0,$$

as required. ■

Here are a couple of examples how one can use Cauchy's integral formula to evaluate the integrals.

Example 11.2. Evaluate

$$\int_{|w-4|=5}\frac{\cos w}{w}\mathrm{d}w.$$

Since 0 is inside the circle with the center 4 and radius 5, I get

$$\int_{|w-4|=5}\frac{\cos w}{w}\mathrm{d}w=2\pi\mathrm{i}f(0)=2\pi\mathrm{i}\cos 0=2\pi\mathrm{i}.$$

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Example 11.3. Evaluate

$$\int_{|w-\mathrm{i}|=1}\frac{w^2}{w^2+1}\mathrm{d}w.$$

Again, the expression under the integral sign is not holomorphic inside $|w-\mathrm{i}|=1$ and hence I cannot conclude that the integral is zero. I can write, however, that

$$\int_{|w-\mathrm{i}|=1}\frac{w^2}{w^2+1}\mathrm{d}w=\int_{|w-\mathrm{i}|=1}\frac{w^2}{(w+\mathrm{i})(w-\mathrm{i})}\mathrm{d}w=\int_{|w-\mathrm{i}|=1}\frac{f(w)}{(w-\mathrm{i})}\mathrm{d}w=2\pi\mathrm{i}f(\mathrm{i})=2\pi\mathrm{i}\frac{\mathrm{i}^2}{\mathrm{i}+\mathrm{i}}=-\pi.$$

Example 11.4. Evaluate

$$\int_{|z|=2}\frac{e^{\frac{\mathrm{i}\pi w}{2}}}{w^2-1}\mathrm{d}w.$$

Here I have a problem that the expression under the integral sign has two problem inside the circle $|z|=2$, at ± 1. To deal with it, I use partial fractions to write

$$\frac{e^{\frac{\mathrm{i}\pi w}{2}}}{w^2-1}=\frac{1}{2}\frac{e^{\frac{\mathrm{i}\pi w}{2}}}{w-1}-\frac{1}{2}\frac{e^{\frac{\mathrm{i}\pi w}{2}}}{w+1}.$$

Therefore, my integral is given now as a sum of two integrals, to each of which Cauchy's formula can be applied, and I get

$$\int_{|z|=2}=2\pi\mathrm{i}\left(e^{\frac{\mathrm{i}\pi\cdot 1}{2}}-e^{\frac{\mathrm{i}\pi\cdot(-1)}{2}}\right)=\mathrm{i}.$$

Cauchy's formula can be used to show that if function f is holomorphic in E, then f' is holomorphic. Using induction it means that if f is holomorphic (differentiable), it is infinitely differentiable! Here is how I can do this.

Proposition 11.5. *Let f be holomorphic on simply connected domain E. Then for any closed positively oriented path $\gamma\subseteq E$ and z inside γ, I have*

$$f'(z)=\frac{1}{2\pi\mathrm{i}}\int_\gamma\frac{f(w)}{(w-z)^2}\mathrm{d}w, \tag{11.1}$$

$$f''(z)=\frac{2!}{2\pi\mathrm{i}}\int_\gamma\frac{f(w)}{(w-z)^3}\mathrm{d}w, \tag{11.2}$$

or in general

$$f^{(n)}(z)=\frac{n!}{2\pi\mathrm{i}}\int_\gamma\frac{f(w)}{(w-z)^{n+1}}\mathrm{d}w. \tag{11.3}$$

I will prove only (11.1), leaving the rest as a similar, but somewhat tedious exercise in performing estimates with complex integrals. Note that equations (11.1) and (11.2) are of very different nature in that that existence of $f'(z)$ is assumed, whereas for (11.2) it must be proved. Moreover, since f'' exists, it means that f' is holomorphic, and induction means that f is infinitely differentiable.

Proof. Again, as in Cauchy's formula, γ can be replaced with $\gamma_\epsilon = \partial B(z,\epsilon)$. Using Cauchy's formula, I can write

$$\frac{f(z+h)-f(z)}{h} = \frac{1}{2\pi \mathrm{i}}\int_{\gamma_\epsilon} f(w)\left(\frac{1}{w-z-h}-\frac{1}{w-z}\right)\mathrm{d}w = \frac{1}{2\pi \mathrm{i}}\int_{\gamma_\epsilon}\frac{f(w)\mathrm{d}w}{(w-z-h)(w-z)}\,.$$

Therefore,

$$\frac{f(z+h)-f(z)}{h} - \frac{1}{2\pi \mathrm{i}}\int_{\gamma_\epsilon}\frac{f(w)}{(w-z)^2}\mathrm{d}w = \frac{h}{2\pi \mathrm{i}}\int_{\gamma_\epsilon}\frac{f(w)\mathrm{d}w}{(w-z-h)(w-z)^2},$$

and the goal now to show that the last expression tends to 0 as $h\to 0$. I choose h such that $|h|<\epsilon/2$ In this case, $|w-z-h|\geq |w-z|-|h|>\epsilon/2$ for all $w\in B(z,\epsilon)$. Since f is holomorphic, it is continuous on γ, and hence bounded, $|f(w)|\leq M$ for $w\in\gamma$. Now by the ML-inequality, I obtain the estimate

$$\left|\frac{f(z+h)-f(z)}{h} - \frac{1}{2\pi \mathrm{i}}\int_{\gamma_\epsilon}\frac{f(w)}{(w-z)^2}\mathrm{d}w\right| \leq M\frac{|h|}{2\pi}\frac{2}{\epsilon^3}\,,$$

which approaches 0 as $h\to 0$.

I will leave it as an exercise to prove the rest of the proposition. ■

Example 11.6. To illustrate applications of the proven proposition to calculations of integrals, consider the following example:

$$\int_{|z|=2}\frac{\cosh z}{(z+1)^3(z-1)}\mathrm{d}z.$$

Function $\cosh z/((z+1)^3(z-1))$ is clearly holomorphic anywhere except two points $z_1=-1, z_2=1$, and both of these points are inside the circle $|z|\leq 2$.

First approach. As before, I will start with using partial fraction decomposition. I find that

$$\frac{1}{(z+1)^3(z-1)} = \frac{1}{8}\cdot\frac{1}{z-1} - \frac{1}{8}\cdot\frac{1}{z+1} - \frac{1}{4}\frac{1}{(z+1)^2} - \frac{1}{2}\frac{1}{(z+1)^3}\,.$$

Therefore, by the linearity of the integral, I need to evaluate four integrals. Using Cauchy's formula for the first two and (11.3) for the last two I find

$$\begin{aligned}
\int_{|z|=2}\frac{\cosh z}{z-1}\mathrm{d}z &= 2\pi\mathrm{i}\cosh 1,\\
\int_{|z|=2}\frac{\cosh z}{z+1}\mathrm{d}z &= 2\pi\mathrm{i}\cosh 1,\\
\int_{|z|=2}\frac{\cosh z}{(z+1)^2}\mathrm{d}z &= 2\pi\mathrm{i}(\cosh z)'\big|_{z=-1} = -2\pi\mathrm{i}\sinh 1,\\
\int_{|z|=2}\frac{\cosh z}{(z+1)^3}\mathrm{d}z &= \frac{2\pi\mathrm{i}}{2!}(\cosh z)''\Big|_{z=-1} = \pi\mathrm{i}\cosh 1,
\end{aligned}$$

and therefore

$$\int_{|z|=2}\frac{\cosh z}{(z+1)^3(z-1)}\mathrm{d}z = \frac{2\pi\mathrm{i}\cosh 1}{8} - \frac{2\pi\mathrm{i}\cosh 1}{8} + \frac{2\pi\mathrm{i}\sinh 1}{4} - \frac{\pi\mathrm{i}\cosh 1}{2} = \frac{\sinh 1-\cosh 1}{2}\pi\mathrm{i} = -\frac{\pi\mathrm{i}}{2e}\,.$$

Second approach. Here I will present a technique that soon will be generalized and put in the general context. First, I connect two sides of my circle $|z| = 2$ with a line segment such that z_1 and z_2 lay in different parts of the circle. Let γ_1 be the closed path composed of the part of the circle and my line segment that encloses z_1 and γ_2 be the closed path composed of the part of the circle and the line segment that encloses z_2 (make a sketch!), and both paths are positively oriented. Due to the properties of the integrals I have

$$\int_{|z|=2} = \int_{\gamma_1} + \int_{\gamma_2} .$$

Now note that each of the integrals on the right hand side has only one "problem" point inside, and to each of them formula (11.3) can be applied. Specifically,

$$\int_{\gamma_1} \frac{\cosh z}{(z+1)^3(z-1)} \mathrm{d}z = \int_{\gamma_1} \frac{\frac{\cosh z}{z-1}}{(z+1)^3} \mathrm{d}z = \frac{2\pi \mathrm{i}}{2!} \left(\frac{\cosh z}{z-1} \right)'' \Big|_{z=-1} = -\frac{2e^{-1} + \cosh 1}{4} \pi \mathrm{i}.$$

$$\int_{\gamma_2} \frac{\cosh z}{(z+1)^3(z-1)} \mathrm{d}z = \int_{\gamma_2} \frac{\frac{\cosh z}{(z+1)^3}}{z-1} \mathrm{d}z = 2\pi \mathrm{i} \frac{\cosh z}{(z+1)^3} \Big|_{z=1} = \pi \mathrm{i} \frac{\cosh 1}{4} .$$

Summing these two results leads to the same answer $-\pi \mathrm{i}/(2e)$.

There are a number of consequences of the proven proposition. One of them is the so-called *Morera's theorem.*

Theorem 11.7. *Let f be continuous in E and*

$$\int_\gamma f = 0$$

for all closed paths $\gamma \subseteq E$. Then f is holomorphic in E.

Proof. Since all the integrals around closed paths are zero, it implies the existence of antiderivative F, which is holomorphic, i.e., $F' = f$. Since by the proven proposition holomorphic function has a holomorphic derivative, f is holomorphic. ■